ON THE PRINCIPAL SYMBOLS OF $K_{\mathbb{C}}$-INVARIANT DIFFERENTIAL OPERATORS ON HERMITIAN SYMMETRIC SPACES

TAKASHI HASHIMOTO

ABSTRACT. Let (G, K) be one of the following classical irreducible Hermitian symmetric pairs of noncompact type: $(\mathrm{SU}(p,q), S(\mathrm{U}(p)\times\mathrm{U}(q)))$, $(\mathrm{Sp}(n,\mathbb{R}), \mathrm{U}(n))$, or $(\mathrm{SO}^*(2n), \mathrm{U}(n))$. Let $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ be complexifications of G and K, respectively, and let P be a maximal parabolic subgroup of $G_{\mathbb{C}}$ whose Levi subgroup is $K_{\mathbb{C}}$. Let V be the holomorphic part of the complexifiaction of the tangent space at the origin of G/K. It is well known that the ring of $K_{\mathbb{C}}$-invariant differential operators on V has a generating system $\{\Gamma_k\}$ given in terms of determinant or Pfaffian that plays an essential rôle in the Capelli identities ([HU91]). Our main result of this paper is that determinant or Pfaffian of the "moment map" on the holomorphic cotangent bundle of $G_{\mathbb{C}}/P$ provides a generating function for the principal symbols of Γ_k's.

1. INTRODUCTION

Let $V := \mathrm{Alt}_n$ be the vector space consisting of all alternating $n\times n$ complex matrices, and $\mathbb{C}[V]$ the vector space consisting of all polynomial functions on V. Then the complex general linear group GL_n acts on V by

$$g.Z := gZ\,{}^tg \quad (g\in\mathrm{GL}_n, Z\in V), \tag{1.1}$$

from which one can define a representation π of GL_n on $\mathbb{C}[V]$ by

$$\pi(g)f(Z) := f(g^{-1}.Z) \quad (g\in\mathrm{GL}_n,\ f\in\mathbb{C}[V]). \tag{1.2}$$

For $Z = (z_{i,j})_{i,j=1,\dots,n} \in V$, with $z_{j,i} = -z_{i,j}$, let $M := (z_{i,j})_{i,j}$ and $D := (\partial_{i,j})_{i,j}$ be the alternating $n\times n$ matrices whose (i, j)-th entries are given by the multiplication operator $z_{i,j}$ and the derivation $\partial_{i,j} := \partial/\partial z_{i,j}$, respectively. Then the representation $\mathrm{d}\pi$ of $\mathfrak{gl}_n$, the Lie algebra of GL_n, induced from π is given by

$$\mathrm{d}\pi(E_{i,j}) = -\sum_{k=1}^{n} z_{k,j}\partial_{k,i} \quad (i, j = 1, 2, \dots, n) \tag{1.3}$$

where $E_{i,j}$ denotes the matrix unit of size $n\times n$ which is a basis for $\mathfrak{gl}_n$.

Let us denote by $U(\mathfrak{gl}_n)$ and $U(\mathfrak{gl}_n)^{\mathrm{GL}_n}$ the universal enveloping algebra of $\mathfrak{gl}_n$, and its subring consisting of GL_n-invariant elements, respectively. Let us denote by $\mathscr{PD}(V)$ and $\mathscr{PD}(V)^{\mathrm{GL}_n}$ the ring of differential operators on V with polynomial coefficients, and its subring consisting of GL_n-invariant differential operators, respectively. Then the following fact is known:

Date: April 28, 2008.

2000 *Mathematics Subject Classification.* 22E47,17B45.

Key words and phrases. Hermitian symmetric space, $K_{\mathbb{C}}$-invariant differential operator, principal symbol, Capelli identity, twisted moment map.

Theorem ([HU91]). *(1) The ring homomorphism $U(\mathfrak{gl}_n)^{\mathrm{GL}_n} \to \mathscr{PD}(V)^{\mathrm{GL}_n}$ induced canonically from $d\pi$ is surjective.*

(2) For $k = 0, 1, \ldots, \lfloor n/2 \rfloor$[1]*, let*

$$\varGamma_k := \sum_{I \subset [n],\, |I| = 2k} \mathrm{Pf}\,(z_I)\,\mathrm{Pf}\,(\partial_I)\,, \tag{1.4}$$

where the summation is taken over all subsets $I \subset [n] := \{1, 2, \ldots, n\}$ such that its cardinality is $2k$, and z_I, ∂_I denote submatrices of M, D consisting of $z_{i,j}, \partial_{i,j}$ with $i, j \in I$.

Then $\{\varGamma_k\}_{k=0,1,\ldots,\lfloor n/2 \rfloor}$ forms a generating system for $\mathscr{PD}(V)^{\mathrm{GL}_n}$,

In particular, for $k = 0, 1, \ldots, \lfloor n/2 \rfloor$, there exist elements of $U(\mathfrak{gl}_n)^{\mathrm{GL}_n}$ that correspond to $\varGamma_k$ under the homomorphism, which are called *skew Capelli elements*. As the names show, they play an essential rôle in the skew Capelli identity.

Now, following [Ito01, KW02], let us consider an alternating $2n \times 2n$ matrix $\mathbf{\Phi}$ with entry in $\mathscr{PD}(V)$ given as follows:

$$\mathbf{\Phi} := \left[\begin{array}{cccc|cccc}
0 & z_{1,2} & \cdots & z_{1,n} & & & & u \\
-z_{1,2} & 0 & \ddots & \vdots & & & u & \\
\vdots & \ddots & 0 & z_{n-1,n} & & \iddots & & \\
-z_{1,n} & \cdots & -z_{n-1,n} & 0 & u & & & \\
\hline
 & & & -u & 0 & \partial_{n-1,n} & \cdots & \partial_{1,n} \\
 & & \iddots & & -\partial_{n-1,n} & 0 & \ddots & \vdots \\
 & -u & & & \vdots & \ddots & 0 & \partial_{1,2} \\
-u & & & & -\partial_{1,n} & \cdots & -\partial_{1,2} & 0
\end{array}\right], \tag{1.5}$$

where $u \in \mathbb{C}$. Though our original motivation of this work is to understand the skew Capelli elements more deeply, let us focus on the corresponding *commutative* objects i.e. the principal symbols of $\varGamma_k$, which we denote by γ_k in this paper, before we enter the *noncommutative* world. So, it is immediate from the minor summation formula of Pfaffian (see [IW06], or (A.1) below) that the principal symbol $\sigma(\mathrm{Pf}\,(\mathbf{\Phi}))$ of $\mathrm{Pf}\,(\mathbf{\Phi})$ provides a generating function for $\{\gamma_k\}$:

$$\sigma(\mathrm{Pf}\,(\mathbf{\Phi})) = \sum_{k=0}^{\lfloor n/2 \rfloor} u^{n-2k}\gamma_k. \tag{1.6}$$

As for the noncommutative counterpart, we can show that $\mathrm{Pf}\,(\mathbf{\Phi})$ can be expanded in the same way as (1.6), but with the coefficient u^{n-2k} of γ_k replaced by a certain monic polynomial in u of degree $n - 2k$ (see [Hasb]). Therefore, if $\mathbf{\Phi}$ came from $U(\mathfrak{gl}_n) \otimes \mathrm{Mat}_n(\mathbb{C})$, it would follow immediately from the property of Pfaffian with noncommutative entry that $\varGamma_k$'s belong to $U(\mathfrak{gl}_n)^{\mathrm{GL}_n}$. However, it is obvious from (1.3) that there exist no elements of $U(\mathfrak{gl}_n)$ that corresponds to the multiplication operator $z_{i,j}$, nor the derivation $\partial_{i,j}$.

What is the natural reason for considering the matrix $\mathbf{\Phi}$ above (or, its commutative counterpart)?

[1] For $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the greatest integer not exceeding x.

We observe that the action (1.1) of GL_n on $V = \mathrm{Alt}_n$ is the holomorphic part of the complexification of isotropy representation at the origin of Hermitian symmetric space $\mathrm{SO}^*(2n)/\mathrm{U}(n)$. Thus we embed $U(\mathfrak{gl}_n)^{\mathrm{GL}_n}$ into $U(\mathfrak{so}_{2n})^{\mathrm{GL}_n}$ and seek for a generating function for $\{\gamma_k\}$ in the latter, in order to find an answer to the question raised above, where $\mathfrak{so}_{2n}$ denotes the complexification of the Lie algebra of $\mathrm{SO}^*(2n)$.

$$\begin{tikzcd}
U(\mathfrak{gl}_n)^{\mathrm{GL}_n} \arrow[dr, hook] \arrow[dd] & \\
& U(\mathfrak{so}_{2n})^{\mathrm{GL}_n} \arrow[dl] \\
\mathscr{P}\mathscr{D}(V)^{\mathrm{GL}_n} &
\end{tikzcd}$$

The real linear Lie group $\mathrm{SO}^*(2n)$ has irreducible unitary representations, called the holomorphic discrete series representations. Among them, we consider a representation π_λ constructed from a holomorphic character λ of a maximal parabolic subgroup P whose Levi subgroup is GL_n via Borel-Weil theory. Note that the representation space of π_λ is a Hilbert space consisting of holomorphic and square-integrable functions defined on an open subset of V, and that the restriction of π_λ to K-finite part coincides with π given in (1.2) when λ is trivial.

Let $\mathrm{d}\pi_\lambda$ be the differential representation induced from π_λ, which we extend to the one of $\mathfrak{so}_{2n}$ by linearity. Take a basis $\{X_i\}$ for $\mathfrak{so}_{2n}$, and its dual basis $\{X_i^\vee\}$, i.e. the basis for $\mathfrak{so}_{2n}$ satisfying that

$$B(X_i, X_j^\vee) = \delta_{i,j},$$

where B is the nondegenerate bilinear form on $\mathfrak{so}_{2n}$ given by $B(X,Y) := \frac{1}{2}\operatorname{tr}(XY)$. For $X \in \mathfrak{so}_{2n}$ given, we denote by $\sigma(X)$ the principal symbol of the differential operator $\mathrm{d}\pi_\lambda(X)$ substituted $\xi_{i,j}$ for $\partial_{i,j}$.

Now we define an element $\sigma(\boldsymbol{X}) \in \mathbb{C}[z_{i,j}, \xi_{i,j}; 1 \leqslant i < j \leqslant n] \otimes \mathfrak{so}_{2n}$ by

$$\sigma(\boldsymbol{X}) := \sum_i \sigma(X_i^\vee) \otimes X_i.$$

Note that $\sigma(\boldsymbol{X})$ is independent of the basis $\{X_i\}$ chosen and that if λ is trivial then it can be considered the moment map on the holomorphic cotangent bundle $T^*(\mathrm{SO}_{2n}/P)$, where SO_{2n} denotes the analytic subgroup of GL_{2n} corresponding to $\mathfrak{so}_{2n}$ (see §7). We then rewrite $\sigma(\boldsymbol{X})$ using γ_1 and a newly introduced (commutative) indeterminate u, which we denote by $\widetilde{\sigma}(\boldsymbol{X})$. We show that Pfaffian of $\widetilde{\sigma}(\boldsymbol{X})$ provides a generating function for $\{\gamma_k\}$ (Corollary 4.5). The reader will find that the matrix $\boldsymbol{\Phi}$ naturally appears on the way (see Theorem 4.4).

All the setup so far applies to the other two types of classical irreducible Hermitian symmetric pairs of noncompact type $(\mathrm{SU}(p,q), S(\mathrm{U}(p)\times\mathrm{U}(q)))$ and $(\mathrm{Sp}(n,\mathbb{R}), \mathrm{U}(n))$. Let (G,K) be one of the two. Then the differential operators $\varGamma_k$ given above have analogous objects, i.e. $K_{\mathbb{C}}$-invariant differential operators acting on the space of polynomial functions on the holomorphic part of the complexification of the tangent space at the origin of G/K which also play an essential rôle in the corresponding Capelli identity, where $K_{\mathbb{C}}$ denotes a complexification of K. In these cases, they are given in terms of

the sum of the product of (minor-)determinants of matrices defined analogously to M and D given above (see (3.1) for their precise definitions). If we define $\widetilde{\sigma}(\boldsymbol{X})$ similarly to the $\mathfrak{so}_{2n}$ case, we can show that determinant of $\widetilde{\sigma}(\boldsymbol{X})$ provides a generating function for their principal symbols (Corollaries 5.4 and 6.4).

The contents of this paper are as follows: In Section 2, we give realizations of $G = \mathrm{SU}(p,q), \mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{SO}^*(2n)$, and their complexification $G_{\mathbb{C}}$. Then we make a brief review of construction of the holomorphic discrete series representations π_λ, via Borel-Weil theory, which are induced from a character λ of a maximal parabolic subgroup of $G_{\mathbb{C}}$ whose Levi subgroup is $K_{\mathbb{C}}$. In Section 3, we recall the definition of the $K_{\mathbb{C}}$-invariant differential operators from [IU01], and introduce our main objects $\sigma(\boldsymbol{X})$ and $\widetilde{\sigma}(\boldsymbol{X})$. More concrete definitions of these objects will be given in the subsequent sections case-by-case. Then in Sections 4, 5 and 6, according as $G = \mathrm{SO}^*(2n), \mathrm{Sp}(n,\mathbb{R})$, or $\mathrm{SU}(p,q)$ $(p \geqslant q)$, we explicitly calculate the differential operators $\mathrm{d}\pi_\lambda(X_i)$ for a basis $\{X_i\}$ of $\mathfrak{g} := \mathrm{Lie}\,(G) \otimes_{\mathbb{R}} \mathbb{C}$, define $\sigma(\boldsymbol{X})$ and $\widetilde{\sigma}(\boldsymbol{X})$ explicitly, and prove our main results that Pfaffian or determinant of $\widetilde{\sigma}(\boldsymbol{X})$ provides a generating function for the principal symbol of the $K_{\mathbb{C}}$-invariant differential operators. In the appendix, we collect some minor summation formulae of Pfaffian and determinant which we make use of in proving the main results.

2. Holomorphic Discrete Series

Henceforth, let G denote one of $\mathrm{SU}(p,q)$ $(p \geqslant q)$, $\mathrm{Sp}(n,\mathbb{R})$, or $\mathrm{SO}^*(2n)$, which we realize as follows:

$$\begin{aligned} \mathrm{SU}(p,q) &= \{g \in \mathrm{SL}_{p+q}(\mathbb{C}); {}^t\overline{g} I_{p,q} g = I_{p,q}\}, \\ \mathrm{Sp}(n,\mathbb{R}) &= \{g \in \mathrm{SU}(n,n); {}^t g J_{n,n} g = J_{n,n}\}, \\ \mathrm{SO}^*(2n) &= \{g \in \mathrm{SU}(n,n); {}^t g J_{2n} g = J_{2n}\}. \end{aligned} \tag{2.1}$$

Here, for positive integers $p, q, n = 1, 2, \ldots$, the matrices $I_{p,q}$, J_n, and $J_{n,n}$ are given by

$$I_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}, \quad J_n = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}, \quad J_{n,n} = \begin{bmatrix} & J_n \\ -J_n & \end{bmatrix},$$

respectively. Let K be a maximal compact subgroup of G given by $K := \{\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right] \in G\}$, where, for an element $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$ in K, the submatrices a and d are of size $p \times p$ and $q \times q$, respectively when $G = \mathrm{SU}(p,q)$, or, are both of size $n \times n$ when $G = \mathrm{SO}^*(2n)$ and $\mathrm{Sp}(n,\mathbb{R})$. Let $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ be the complexifications of G and K, respectively, and P the maximal parabolic subgroup of $G_{\mathbb{C}}$ given by $P := \{\left[\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right] \in G_{\mathbb{C}}\}$ so that its Levi subgroup equals $K_{\mathbb{C}}$. Define a holomorphic character $\lambda : P \to \mathbb{C}^\times$ by

$$\lambda(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}) = (\det d)^{-s} \tag{2.2}$$

for $s \in \mathbb{Z}$. Denote by $\mathbb{C}_\lambda$ the one-dimensional P-module defined by $p.v = \lambda(p)v$ $(p \in P, v \in \mathbb{C})$, and by L_λ the pull-back of the holomorphic line bundle $G_{\mathbb{C}} \times_P \mathbb{C}_\lambda$ by the embbeding $G/K = GP/P \hookrightarrow G_{\mathbb{C}}/P$.

The space $\Gamma(L_\lambda)$ of all holomorphic sections for L_λ are identified with the space of all holomorphic functions f on GP, which is an open subset of $G_{\mathbb{C}}$, that satisfy

$$f(xp) = \lambda(p)^{-1} f(x) \quad (x \in GP, p \in P). \tag{2.3}$$

Define $\Gamma^2(L_\lambda)$ to be the subspace of $\Gamma(L_\lambda)$ consisting of square integrable holomorphic sections with respect to Haar measure on G:

$$\Gamma^2(L_\lambda) := \{ f \in \Gamma(L_\lambda); \int_G |f(g)|^2 \, dg < \infty \}, \tag{2.4}$$

and an action T_λ of G on $\Gamma^2(L_\lambda)$ by

$$(T_\lambda(g) f)(x) := f(g^{-1}x) \quad (g \in G, x \in GP).$$

Then $(T_\lambda, \Gamma^2(L_\lambda))$ is an irreducible unitary representation of G, called the *holomorphic discrete series* (if it is not zero). One must impose some condition on s in order to require that $\Gamma^2(L_\lambda)$ be nonzero, which we do not consider here since we will be concerned with the representations of Lie algebras in this paper (see e.g. [Kna86] for the condition).

Now, according as $G = \mathrm{SU}(p,q)$ $(p \geqslant q), \mathrm{Sp}(n,\mathbb{R})$, or $\mathrm{SO}^*(2n)$, let us realize the bounded symmetric domain Ω as follows:

if $G = \mathrm{SU}(p,q)$, $\quad \Omega := \{ z \in \mathrm{Mat}_{p,q}(\mathbb{C}); 1_q - {}^t\overline{z}z > 0 \}$;
if $G = \mathrm{Sp}(n,\mathbb{R})$, $\quad \Omega := \{ z \in \mathrm{Mat}_n(\mathbb{C}); 1_n - {}^t\overline{z}z > 0, J_n {}^tz J_n = z \}$;
if $G = \mathrm{SO}^*(2n)$, $\quad \Omega := \{ z \in \mathrm{Mat}_n(\mathbb{C}); 1_n - {}^t\overline{z}z > 0, J_n {}^tz J_n = -z \}$,

and let G act on Ω by linear fractional transformation:

$$g.z = (az+b)(cz+d)^{-1} \quad (g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G, z \in \Omega). \tag{2.5}$$

Then Ω is isomorphic to G/K in each case. If we denote by $\mathscr{O}(\Omega)$ the space of all holomorphic functions on Ω, define a map

$$\Phi : \Gamma(L_\lambda) \to \mathscr{O}(\Omega), \qquad f \mapsto F$$

by

$$F(z) = f(\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}).$$

Then Φ is a bijection. Let $\mathscr{H}_\lambda := \Phi(\Gamma^2(L_\lambda))$, and define an action π_λ of G on $\mathscr{H}_\lambda$ so that the diagram (2.6) commutes for all $g \in G$:

$$\begin{array}{ccc} \Gamma^2(L_\lambda) & \xrightarrow{\Phi} & \mathscr{H}_\lambda \\ {\scriptstyle T_\lambda(g)}\downarrow & & \downarrow{\scriptstyle \pi_\lambda(g)} \\ \Gamma^2(L_\lambda) & \xrightarrow{\Phi} & \mathscr{H}_\lambda . \end{array} \tag{2.6}$$

Explicitly, it is given by

$$(\pi_\lambda(g)F)(z) = \det(cz+d)^s \, F((az+b)(cz+d)^{-1}) \tag{2.7}$$

for $g \in G$ and $F \in \mathscr{H}_\lambda$, where $g^{-1} = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$.

Let us introduce a few more notations. Let $\mathfrak{g} := \mathrm{Lie}\,(G) \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{p} := \mathrm{Lie}\,(P)$, $\overline{\mathfrak{u}}$ the nilradical of $\mathfrak{p}$, and $\mathfrak{u}$ its opposite. Then we have $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ and $\mathfrak{p} = \mathfrak{k} \oplus \overline{\mathfrak{u}}$, where

$\mathfrak{k} := \mathrm{Lie}\,(K) \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathrm{d}\pi_\lambda$ be the differential representation of $\mathrm{Lie}\,(G)$ induced from π_λ, which we extend to the one of the complex Lie algebra $\mathfrak{g}$ by linearity. Furthermore, identifying Ω with an open subset of $\mathfrak{u}$ by $z \leftrightarrow \left[\begin{smallmatrix} 0 & z \\ 0 & 0 \end{smallmatrix}\right]$, let U_Ω be the image of $\Omega \subset \mathfrak{u}$ by the exponential map.

Since we realize the real linear Lie groups G as in (2.1), the corresponding complex Lie algebras $\mathfrak{g}$ are given by

$$\begin{aligned} \mathfrak{sl}_{p+q} &= \{X \in \mathrm{Mat}_{p+q}(\mathbb{C});\, \mathrm{tr}\,(X) = 0\}, \\ \mathfrak{sp}_n &= \{X \in \mathrm{Mat}_{2n}(\mathbb{C});\, {}^tXJ_{n,n} + XJ_{n,n} = O\}, \\ \mathfrak{so}_{2n} &= \{X \in \mathrm{Mat}_{2n}(\mathbb{C});\, {}^tXJ_{2n} + XJ_{2n} = O\}, \end{aligned} \tag{2.8}$$

respectively.

Remark 2.1. For $X \in \mathfrak{g}$, if $t \in \mathbb{R}$ is sufficiently small then $\exp tX$ acts on Ω by (2.5), hence on $\mathscr{H}_\lambda$ by (2.7), so that its differential at $t = 0$ coincides with $\mathrm{d}\pi_\lambda$.

3. Principal Symbols of $K_{\mathbb{C}}$-invariant Differential Operators

Let V denote the holomorphic part of the tangent space at the origin of G/K. Then one can identify V with $\mathfrak{u}$ and construct a representation of $K_{\mathbb{C}}$ on the space $\mathbb{C}[V]$ of all polynomial function on V through the action of $K_{\mathbb{C}}$ on $V = \mathfrak{u}$.

Let $\mathscr{P}\mathscr{D}(V)^{K_{\mathbb{C}}}$ be the ring of $K_{\mathbb{C}}$-invariant differential operators with polynomial coefficient and let $r := \mathbb{R}\text{-}\operatorname{rank} G$, the real rank of G. Then it is well known that there exists a generating system $\{\Gamma_k\}_{k=0,1,\ldots,r}$ for $\mathscr{P}\mathscr{D}(V)^{K_{\mathbb{C}}}$ which are given in terms of determinant or Pfaffian as follows:

(1) For $G = \mathrm{SU}(p, q)$ with $p \geqslant q$,

$$\Gamma_k = \sum_{\substack{I\subset[p],J\subset[q] \\ |I|=|J|=k}} \det(z^I_J) \det(\partial^I_J), \quad (k = 0, 1, \ldots, q); \tag{3.1a}$$

(2) For $G = \mathrm{Sp}(n, \mathbb{R})$,

$$\Gamma_k = \sum_{\substack{I,J\subset[n] \\ |I|=|J|=k}} \det(z^I_J) \det(\tilde{\partial}^I_J), \quad (k = 0, 1, \ldots, n); \tag{3.1b}$$

(3) For $G = \mathrm{SO}^*(2n)$,

$$\Gamma_k = \sum_{I\subset[n],|I|=2k} \mathrm{Pf}\,(z_I)\,\mathrm{Pf}\,(\partial_I)\,, \quad (k = 0, 1, \ldots, \lfloor n/2 \rfloor) \tag{3.1c}$$

([HU91]; see below for details). We will find a generating function for the principal symbols of the differential operators Γ_k.

Define a $G_{\mathbb{C}}$-invariant nondegenerate bilinear form B on $\mathfrak{g}$ by

$$B(X, Y) = \begin{cases} \mathrm{tr}\,(XY) & \text{if } \mathfrak{g} = \mathfrak{sl}_{p+q}, \\ \frac{1}{2}\,\mathrm{tr}\,(XY) & \text{if } \mathfrak{g} = \mathfrak{so}_{2n} \text{ or } \mathfrak{sp}_n. \end{cases} \tag{3.2}$$

Given a basis $\{X_i\}_{i=1,\ldots,\dim\mathfrak{g}}$ for $\mathfrak{g}$, let us denote the dual basis with respect to B by $\{X_i^\vee\}$, i.e. the basis for $\mathfrak{g}$ satisfying

$$B(X_i, X_j^\vee) = \delta_{i,j}$$

for $i, j = 1, \ldots, \dim \mathfrak{g}$. Then we define an element $\boldsymbol{X}$ of $U(\mathfrak{g}) \otimes \mathrm{Mat}_N(\mathbb{C})$ by

$$\boldsymbol{X} := \sum_{i=1}^{\dim \mathfrak{g}} X_i^\vee \otimes X_i, \tag{3.3}$$

where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. Namely, we regard the former and the latter factors as elements of $U(\mathfrak{g})$ and $\mathrm{Mat}_N(\mathbb{C})$, respectively, where $N = p + q$ if $\mathfrak{g} = \mathfrak{sl}_{p+q}$, and $N = 2n$ if $\mathfrak{g} = \mathfrak{sp}_n$ or $\mathfrak{so}_{2n}$.

Denoting by $\sigma(X)$ the principal symbol of $d\pi_\lambda(X)$ for $X \in \mathfrak{g}$, let us define a $\mathfrak{g}$-valued polynomial function on the holomorphic cotangent bundle $T^*(G_{\mathbb{C}}/P)$ by

$$\sigma(\boldsymbol{X}) := \sum_{i=1}^{\dim \mathfrak{g}} \sigma(X_i^\vee) \otimes X_i. \tag{3.4}$$

By definition, $\boldsymbol{X}$ and $\sigma(\boldsymbol{X})$ are independent of the basis $\{X_i\}$ chosen.

As we will explain below in more detail, if the principal symbol $\sigma(X_i^\vee)$ contains the parameter s, we rewrite it using γ_1, the principal symbol of the Euler operator $\mathit{\Gamma}_1$ on V, and introduce a new indeterminate u; then we substitute $s - u$ into $\gamma_1$in $\sigma(X_i^\vee)$ which we denote by $\widetilde{\sigma}(X_i^\vee)$. If the principal symbol $\sigma(X_i^\vee)$ does not contain s, we set $\widetilde{\sigma}(X_i^\vee) := \sigma(X_i^\vee)$. Finally, we define

$$\widetilde{\sigma}(\boldsymbol{X}) := \sum_{i=1}^{\dim \mathfrak{g}} \widetilde{\sigma}(X_i^\vee) \otimes X_i. \tag{3.5}$$

We will show that determinant or Pfaffian of $\widetilde{\sigma}(\boldsymbol{X})$ yields a generating function for the principal symbols $\{\gamma_k\}_{k=0,1,\ldots,r}$ of the generators $\{\mathit{\Gamma}_k\}$ mentioned above in each case of (1), (2), and (3) in the following sections.

4. The Case $G = \mathrm{SO}^*(2n)$

First let us consider the case where $G = \mathrm{SO}^*(2n)$, or $\mathfrak{g} = \mathfrak{so}_{2n}$. In this section and the next, $E_{i,j}$ denotes the $2n \times 2n$ matrix with its (i, j)-th entry being 1 and all the others 0. Write $Z \in V = \mathfrak{u}$ as $Z = \sum_{i,j \in [n]} z_{i,j} E_{i,-j}$, $z_{j,i} = -z_{i,j}$, where we agree that $-i$ stands for $2n + 1 - i$. Let M and D denote alternating matrices of size n whose (i, j)-th entry are given by the multiplication operators $z_{i,j}$ and the differential operators $\partial_{i,j} := \partial/\partial z_{i,j}$, respectively. Then

$$\mathit{\Gamma}_k := \sum_{I \subset [n],\, |I| = 2k} \mathrm{Pf}\,(z_I)\,\mathrm{Pf}\,(\partial_I) \qquad (k = 0, 1, \ldots, \lfloor n/2 \rfloor)$$

form a generating system for $\mathscr{PD}(V)^{K_{\mathbb{C}}}$ with $K_{\mathbb{C}} = \mathrm{GL}_n$, where z_I, ∂_I denotes submatrices of M, D consisting of the entries whose row- and column- indices are both in $I \subset [n]$ ([HU91]).

Take a basis $\{X_{i,j}^\epsilon\}_{\epsilon = 0.\pm; i,j \in [n]}$ for $\mathfrak{so}_{2n}$ as follows:

$$\begin{aligned} X_{i,j}^0 &:= E_{i,j} - E_{-j,-i} \quad (1 \leqslant i, j \leqslant n) \\ X_{i,j}^+ &:= E_{i,-j} - E_{j,-i} \quad (1 \leqslant i < j \leqslant n) \\ X_{i,j}^- &:= E_{-j,i} - E_{-i,j} \quad (1 \leqslant i < j \leqslant n) \end{aligned} \tag{4.1}$$

Proposition 4.1. *The differential operators* $\mathrm{d}\pi_\lambda(X^{\epsilon}_{i,j})$ $(\epsilon = 0, \pm; i, j \in [n])$ *are given by*

$$\mathrm{d}\pi_\lambda(X^0_{i,j}) = s\delta_{i,j} - \sum_{k\in[n]} z_{k,j}\partial_{k,i}, \tag{4.2}$$

$$\mathrm{d}\pi_\lambda(X^+_{i,j}) = -\partial_{i,j}, \tag{4.3}$$

$$\mathrm{d}\pi_\lambda(X^-_{i,j}) = -2sz_{i,j} - \sum_{1\leqslant k<l\leqslant n} \left(z_{k,i}z_{j,l} - z_{k,j}z_{i,l}\right)\partial_{k,l}. \tag{4.4}$$

Proof. Let us denote the $n\times n$ matrix whose (i, j)-th entry is 1 and all the others are 0 by $E^{(n)}_{i,j}$, and let $\tilde{z} = \sum_{k.l\in[n]} z_{k,l}E^{(n)}_{k,l}$ with $z_{l,k} = -z_{k,l}$. Then $z := \tilde{z}J_n$ belongs to Ω if it is positive definite.

(I) First we calculate $\mathrm{d}\pi_\lambda(X^0_{ij})$. Writing $\left[\begin{smallmatrix} a & 0\\ 0 & d\end{smallmatrix}\right] := \exp(-tX^0_{ij})$, we see that

$$\begin{aligned}
\exp(-tX^0_{ij}).z &= azd^{-1}\\
&= \left(1_n - tE^{(n)}_{i,j} + O(t^2)\right) z \left(1_n + tE^{(n)}_{n+1-j,n+1-i} + O(t^2)\right)^{-1}\\
&= \left(\tilde{z} - t(E^{(n)}_{i,j}\tilde{z} + \tilde{z}E^{(n)}_{j,i})\right) J_n + O(t^2)\\
&= \Big(\sum_{k,l\in[n]} z_{k,l}E^{(n)}_{k,l} - t\sum_{k,l\in[n]} z_{k,l}(E^{(n)}_{i,j}E^{(n)}_{k,l} + E^{(n)}_{k,l}E^{(n)}_{j,i})\Big)J_n + O(t^2)\\
&= \sum_{k,l\in[n]} \left(z_{k,l} - t(\delta_{i,k}z_{j,l} + \delta_{i,l}z_{k,j})\right) E^{(n)}_{k,l}J_n + O(t^2)
\end{aligned}$$

and that

$$(\det d)^s = \left(\det(1_n + tE^{(n)}_{n+1-j,n+1-i} + O(t^2))\right)^s = e^{st\,\mathrm{tr}\left(E^{(n)}_{n+1-j,n+1-i}\right)+O(t^2)}.$$

Therefore, for $F \in \mathscr{H}_\lambda$, we obtain that

$$\begin{aligned}
(\mathrm{d}\pi_\lambda(X^0_{ij})F)(z) &= \left.\frac{d}{dt}\right|_{t=0} (\pi_\lambda(\exp(tX^0_{ij}))F)(z)\\
&= \left.\frac{d}{dt}\right|_{t=0} e^{st\,\mathrm{tr}\left(E^{(n)}_{n+1-j,n+1-i}\right)}F\Big(\sum_{k,l\in[n]}(z_{k,l} - t(\delta_{i,k}z_{j,l} + \delta_{i,l}z_{k,j}))E^{(n)}_{k,l}J_n\Big)\\
&= \Big(s\delta_{i,j} - \sum_{k<l}(\delta_{i,k}z_{j,l} + \delta_{i,l}z_{k,j})\partial_{k,l}\Big)F(z)\\
&= \Big(s\delta_{i,j} - \sum_{i<l} z_{j,l}\partial_{i,l} - \sum_{k<i} z_{k,j}\partial_{k,i}\Big)F(z)\\
&= \Big(s\delta_{i,j} - \sum_{k\in[n]} z_{j,k}\partial_{i,k}\Big)F(z).
\end{aligned}$$

(II) Next we calculate $\mathrm{d}\pi_\lambda(X^+_{ij})$. Writing $\left[\begin{smallmatrix} 1 & b\\ 0 & 1\end{smallmatrix}\right] := \exp(-tX^+_{ij})$, we see that

$$\begin{aligned}
\exp(-tX^+_{ij}).z &= z + b\\
&= (\tilde{z} - t(E^{(n)}_{i,j} - E^{(n)}_{j,i}))J_n\\
&= \sum_{k,l\in[n]} \left(z_{k,l} - t\delta_{i,k}\delta_{j,l} + t\delta_{j,k}\delta_{i,l}\right) E^{(n)}_{k,l}J_n,
\end{aligned}$$

and hence obtain that

$$\begin{aligned} d\pi_\lambda(X^+_{ij}) &= -\sum_{k<l}(\delta_{i,k}\delta_{j,l} - \delta_{i,l}\delta_{j,k})\partial_{k,l} \\ &= -\partial_{i,j}. \end{aligned}$$

(III) Finally, let us calculate $d\pi_\lambda(X^-_{ij})$. Writing $\left[\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right] := \exp(-tX^-_{ij})$, we see that

$$\begin{aligned} \exp(-tX^-_{ij}).z &= z(1+cz)^{-1} \\ &= z\left(1_n + tJ_n(E^{(n)}_{i,j} - E^{(n)}_{j,i})z\right)^{-1} \\ &= \left(\tilde{z} - t\tilde{z}(E^{(n)}_{i,j} - E^{(n)}_{j,i})\tilde{z}\right) J_n + O(t^2) \\ &= \left(\sum_{k,l} z_{k,l}E^{(n)}_{k,l} - t\sum_{a,b,c,d} z_{a,b}z_{c,d}E^{(n)}_{a,b}(E^{(n)}_{i,j} - E^{(n)}_{j,i})E^{(n)}_{c,d}\right)J_n + O(t^2) \\ &= \sum_{k,l}\left(z_{k,l} - t(z_{k,i}z_{j,l} - z_{k,j}z_{i,l})\right)E^{(n)}_{k,l}J_n + O(t^2), \end{aligned}$$

and that

$$\begin{aligned} \det(1+cz)^s &= \left(\det(1_n + tJ_n(E^{(n)}_{i,j} - E^{(n)}_{j,i})\tilde{z}J_n)\right)^s = e^{st\,\mathrm{tr}\left((E^{(n)}_{i,j} - E^{(n)}_{j,i})\tilde{z}\right) + O(t^2)} \\ &= e^{st(z_{j,i} - z_{i,j}) + O(t^2)} = e^{-2stz_{i,j} + O(t^2)}, \end{aligned}$$

from which we obtain that

$$d\pi_\lambda(X^-_{ij}) = -2sz_{i,j} - \sum_{k<l}\left(z_{k,i}z_{j,l} - z_{k,j}z_{i,l}\right)\partial_{k,l}.$$

This completes the proof. □

Noting that the dual basis of (4.1) is given by

$$(X^0_{i,j})^\vee = X^0_{j,i}, \qquad\qquad (X^\pm_{i,j})^\vee = X^\mp_{i,j},$$

let us define the element $\boldsymbol{X}$ of $U(\mathfrak{g}) \otimes \mathrm{Mat}_{2n}(\mathbb{C})$ by (3.3); it looks like

$$\boldsymbol{X} = \left[\begin{array}{cccc|cccc} X^0_{1,1} & X^0_{2,1} & \cdots & X^0_{n,1} & X^-_{1,n} & \cdots & X^-_{1,2} & 0 \\ X^0_{1,2} & X^0_{2,2} & \cdots & X^0_{n,2} & \vdots & \iddots & 0 & -X^-_{1,2} \\ \vdots & \vdots & & \vdots & X^-_{n-1,n} & 0 & \iddots & \vdots \\ X^0_{1,n} & X^0_{2,n} & \cdots & X^0_{n,n} & 0 & -X^-_{n-1,n} & \cdots & -X^-_{1,n} \\ \hline X^+_{n,1} & \cdots & X^+_{n-1,n} & 0 & -X^0_{n,n} & \cdots & -X^0_{n,2} & -X^0_{n,1} \\ \vdots & \iddots & 0 & -X^+_{n-1,n} & \vdots & & \vdots & \vdots \\ X^+_{1,2} & 0 & \iddots & \vdots & -X^0_{2,n} & \cdots & -X^0_{2,2} & -X^0_{2,1} \\ 0 & -X^+_{1,2} & \cdots & -X^+_{1,n} & -X^0_{1,n} & \cdots & -X^0_{1,2} & -X^0_{1,1} \end{array}\right].$$

Remark 4.2. For the matrix $\boldsymbol{X}$ given above, it is well known that Pfaffian[2] $\mathrm{Pf}\,(\boldsymbol{X})$ is a central element of the universal enveloping algebra $U(\mathfrak{so}_{2n})$ (see [IU01] or [Hasa] for definition and the properties of Pfaffian with noncommutative entry).

[2]Throughout the paper, for a given $2n \times 2n$ matrix A alternating along the antidiagonal, we denote $\mathrm{Pf}\,(AJ_{2n})$ by $\mathrm{Pf}\,(A)$ for brevity.

Following the prescription (3.4), let us define $\sigma(\boldsymbol{X})$ by substituting $\xi_{i,j}$ into $\partial_{i,j}$:

$$\sigma(\boldsymbol{X}) := \sum_{\epsilon;i,j} \sigma((X^{\epsilon}_{i,j})^{\vee}) \otimes X^{\epsilon}_{i,j}. \tag{4.5}$$

Theorem 4.3. *Let $u(z) := \exp \sum_{i<j} z_{i,j} X^{+}_{i,j} \in U_{\Omega}$. Then we have*

$$\mathrm{Ad}(u(z)^{-1})\sigma(\boldsymbol{X}) = s \sum_{i} X^{0}_{i,i} - \sum_{i<j} \xi_{i,j} X^{-}_{i,j} \tag{4.6}$$

$$= \left[\begin{array}{cccc|cccc} s & & & & & & & \\ & s & & & & & & \\ & & \ddots & & & & & \\ & & & s & & & & \\ \hline -\xi_{1,n-1} & -\xi_{2,n-1} & \cdots & 0 & -s & & & \\ \vdots & \vdots & \iddots & \vdots & & \ddots & & \\ -\xi_{1,2} & 0 & \cdots & \xi_{2,n-1} & & & -s & \\ 0 & \xi_{1,2} & \cdots & \xi_{1,n-1} & & & & -s \end{array}\right]. \tag{4.7}$$

Proof. This is just a simple matrix calculation, but we give a rather detailed one, which we will need in proving the theorem stated below. In what follows, for a matrix A given, let us denote the (i, j)-th entry of A by A_{ij}.

Writing $\left[\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right] := u(z)$ and $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] := \sigma(\boldsymbol{X})$, we have

$$\mathrm{Ad}(u(z)^{-1})\sigma(\boldsymbol{X}) = \begin{bmatrix} A - zC & Az - zCz + B - zD \\ C & Cz + D \end{bmatrix} \tag{4.8}$$

(I) First, we calculate the $(1, 1)$ and $(2, 2)$-blocks. Let $\tilde{C} := J_n C$. Note that, by definition, A_{ij} is $\sigma(X^{0}_{j,i})$. Since $(zC)_{ij} = (\tilde{z}\tilde{C})_{ij}$ equals $\sum_{k=1}^{n} z_{i,k}\xi_{k,j} = -\sum_{k=1}^{n} z_{i,k}\xi_{j,k}$, it follows from (4.2) that

$$A = s1_n + zC. \tag{4.9}$$

Then the fact that (4.8) is an element of $\mathfrak{so}_{2n}$ implies that

$$Cz + D = -J_n\, {}^{t}(A - zC) J_n = -s1_n. \tag{4.10}$$

(II) Next we show that the $(1, 2)$-block equals 0. Let $\tilde{B} := BJ_n$. Since $(\tilde{z}\tilde{C}\tilde{z})_{ij}$ equals

$$\sum_{k,l\in[n]} z_{i,k}\xi_{k,l}z_{l,j} = \left(\sum_{k<l} + \sum_{k>l}\right) z_{i,k}z_{l,j}\xi_{k,l} = \sum_{1\leqslant k<l\leqslant n} (z_{k,i}z_{j,l} - z_{k,j}z_{i,l})\xi_{k,l},$$

it follows from (4.4) that $\tilde{B} = -2s\tilde{z} - \tilde{z}\tilde{C}\tilde{z}$ and

$$B = -2sz - zCz. \tag{4.11}$$

Therefore we obtain that

$$\begin{aligned} Az - zCz + B - zD &= (s1_n + zC)z - zCz - 2sz - zCz - z(-s1_n - Cz) \\ &= 0. \end{aligned}$$

This completes the proof. □

Now, using the principal symbol $\gamma_1 = \sum_{k<l} z_{k,l}\xi_{k,l}$ of the Euler operator Γ_1 on V, we rewrite $\sigma(X^0_{i,i})$ and $\sigma(X^-_{i,j})$, which are the only principal symbols that contain the parameter s, as follows:

$$\sigma(X^0_{i,i}) = s - \gamma_1 + \sum_{\substack{k<l\\ k,l\neq i}} z_{k,l}\xi_{k,l},$$

$$\sigma(X^-_{i,j}) = -2sz_{i,j} + z_{i,j}\gamma_1 - \sum_{\substack{k<l\\ \{k,l\}\cap\{i,j\}=\varnothing}} (z_{i,j}z_{k,l} - z_{k,i}z_{j,l} + z_{k,j}z_{i,l})\xi_{k,l}.$$

Then we substitute $s-u$ into γ_1 with a new indeterminate u in the symbols above, which we denote by $\widetilde{\sigma}(X^0_{i,i})$ and $\widetilde{\sigma}(X^-_{i,j})$, respectively:

$$\widetilde{\sigma}(X^0_{i,i}) := u + \sum_{\substack{k<l\\ k,l\neq i}} z_{k,l}\xi_{k,l}, \tag{4.12}$$

$$\widetilde{\sigma}(X^-_{i,j}) := -(u+s)z_{i,j} - \sum_{\substack{k<l\\ \{k,l\}\cap\{i,j\}=\varnothing}} (z_{i,j}z_{k,l} - z_{i,k}z_{j,l} + z_{i,l}z_{j,k})\xi_{k,l}. \tag{4.13}$$

As for the other symbols, set $\widetilde{\sigma}(X^\epsilon_{i,j}) := \sigma(X^\epsilon_{i,j})$. Then we define $\widetilde{\sigma}(\boldsymbol{X})$ by

$$\widetilde{\sigma}(\boldsymbol{X}) := \sum_{\epsilon;i,j} \widetilde{\sigma}((X^\epsilon_{i,j})^\vee) \otimes X^\epsilon_{i,j}.$$

Theorem 4.4. *Let $u(z) \in U_\Omega$ be as in Theorem 4.3. Then we have*

$$\mathrm{Ad}(u(z)^{-1})\widetilde{\sigma}(\boldsymbol{X}) = (u+\gamma_1)\sum_i X^0_{i,i} - \sum_{i<j}\xi_{i,j}X^-_{i,j} - (s-(u+\gamma_1))\sum_{i<j} z_{i,j}X^+_{i,j} \tag{4.14}$$

$$= \left[\begin{array}{cccc|cccc} u+\gamma_1 & & & & -\tau z_{1,n-1} & \cdots & -\tau z_{1,2} & 0 \\ & u+\gamma_1 & & & -\tau z_{2,n-1} & \cdots & 0 & \tau z_{1,2} \\ & & \ddots & & \vdots & ⋰ & \vdots & \vdots \\ & & & u+\gamma_1 & 0 & \cdots & \tau z_{2,n-2} & \tau z_{1,n-1} \\ \hline -\xi_{1,n-1} & -\xi_{2,n-1} & \cdots & 0 & -u-\gamma_1 & & & \\ \vdots & \vdots & ⋰ & \vdots & & \ddots & & \\ -\xi_{1,2} & 0 & \cdots & \xi_{2,n-1} & & & -u-\gamma_1 & \\ 0 & \xi_{1,2} & \cdots & \xi_{1,n-1} & & & & -u-\gamma_1 \end{array}\right]. \tag{4.15}$$

Here, we set $\tau := s-(u+\gamma_1)$ in (4.15) *for brevity.*

Proof. Let $\left[\begin{smallmatrix} A(u) & B(u) \\ C & D(u) \end{smallmatrix}\right] := \widetilde{\sigma}(\boldsymbol{X})$. (Note that the submatrix C is the same as in the proof of Theorem 4.3). Then it suffices to show that

$$A(u) - zC = (u+\gamma_1)1_n, \tag{4.16}$$

$$A(u)z - zCz + B(u) - zD(u) = (u+\gamma_1-s)z, \tag{4.17}$$

$$Cz + D(u) = -(u+\gamma_1)1_n \tag{4.18}$$

(see (4.8)).

(I) First we show (4.16) and (4.18). But the latter follows from the former, as in the proof of Theorem 4.3. Note that $A(u)_{ij}$ can be written as

$$A(u)_{ij} = (u + \sum_{k<l} z_{k,l}\xi_{k,l})\delta_{i,j} - \sum_{k=1}^{n} z_{k,i}\xi_{k,j}. \tag{4.19}$$

Since the second summation in (4.19) is equal to $-(\tilde{z}\tilde{C})_{ij} = -(zC)_{ij}$ as shown in (I) of the proof of Theorem 4.3, we obtain that

$$A(u) = (u + \gamma_1)1_n + zC.$$

(II) Next we show (4.17). Let $\tilde{B}(u) := B(u)J_n$. Since the summation part of $\widetilde{\sigma}(X^-_{i,j}) = \tilde{B}(u)_{ij}$ equals $z_{i,j}\gamma_1 + (\tilde{z}\tilde{C}\tilde{z})_{ij}$ as in (II) of the proof of Theorem 4.3, we obtain that

$$B(u) = -(u + s + \gamma_1)z - zCz. \tag{4.20}$$

Now, combining (4.20) with (4.16) and (4.18), we obtain that

$$A(u)z - zCz + B(u) - zD(u) = (u + \gamma_1 - s)z.$$

This completes the proof. □

Note the similarity between the matrix $\mathbf{\Phi}$ given in (1.5) and $\widetilde{\sigma}(\boldsymbol{X})$ conjugated by $u(z)^{-1} \in U_\Omega$ given in (4.15). It follows immediately from Theorem 4.4 and the minor summation formula of Pfaffian (A.1) that $\operatorname{Pf}(\widetilde{\sigma}(\boldsymbol{X}))$ yields a generating function for $\{\gamma_k\}$.

Corollary 4.5. *Retain the notation above. Then we have the following formula:*

$$\operatorname{Pf}(\widetilde{\sigma}(\boldsymbol{X})) = \sum_{k=0}^{\lfloor n/2 \rfloor} (u + \gamma_1)^{n-2k}(s - (u + \gamma_1))^k \gamma_k. \tag{4.21}$$

5. The Case $G = \mathrm{Sp}(n, \mathbb{R})$

Next we consider the case where $G = \mathrm{Sp}(n, \mathbb{R})$, or $\mathfrak{g} = \mathfrak{sp}_n$. Let $E_{i,j}$ be as in the previous section. Write an element $Z \in V = \mathfrak{u}$ as $Z = \sum_{i,j\in[n]} z_{i,j}E_{i,-j}$, $z_{j,i} = z_{i,j}$ and let

$$\tilde{\partial}_{i,j} = \begin{cases} 2\partial_{i,j} & (i = j) \\ \partial_{i,j} & (i \neq j). \end{cases}$$

Let us denote by M, D symmetric matrices of size n whose (i, j)-th entry are given by the multiplication operators $z_{i,j}$ and the differential operators $\tilde{\partial}_{i,j}$. Then

$$\Gamma_k := \sum_{\substack{I,J\subset[n] \\ |I|=|J|=k}} \det(z^I_J)\det(\tilde{\partial}^I_J) \quad (k = 0, 1, \ldots, n)$$

form a generating system for $\mathscr{P}\mathscr{D}(V)^{K_\mathbb{C}}$ with $K_\mathbb{C} = \mathrm{GL}_n$, where z^I_J, ∂^I_J denote submatrices of M, D consisting of the entries whose row- and column- indices are in I and J, respectively ([HU91]).

Take a basis $\{X^{\epsilon}_{i,j}\}_{\epsilon=0,\pm;i,j\in[n]}$ for $\mathfrak{sp}_n$ as follows:

$$\begin{aligned} X^0_{i,j} &:= E_{i,j} - E_{-j,-i} \quad (1 \leqslant i, j \leqslant n),\\ X^+_{i,j} &:= E_{i,-j} + E_{j,-i} \quad (1 \leqslant i \leqslant j \leqslant n),\\ X^-_{i,j} &:= E_{-j,i} + E_{-i,j} \quad (1 \leqslant i \leqslant j \leqslant n). \end{aligned} \tag{5.1}$$

Proposition 5.1. *The differential operators* $\mathrm{d}\pi_\lambda(X^\epsilon_{i,j})$ $(\epsilon = 0, \pm; i, j \in [n])$ *are given by*

$$\mathrm{d}\pi_\lambda(X^0_{i,j}) = s\delta_{i,j} - \sum_{k=1}^{n} z_{j,k}\tilde{\partial}_{i,k}, \tag{5.2}$$

$$\mathrm{d}\pi_\lambda(X^+_{i,j}) = -\tilde{\partial}_{i,j}, \tag{5.3}$$

$$\mathrm{d}\pi_\lambda(X^-_{i,j}) = -2sz_{i,j} - \sum_{1\leqslant k,l\leqslant n} z_{k,i}z_{j,l}\tilde{\partial}_{k,l}. \tag{5.4}$$

Proof. Let $E^{(n)}_{i,j}$ be as in the proof of Proposition 4.1, and let $\tilde{z} := \sum_{k,l} z_{k,l}E^{(n)}_{k,l}$ with $z_{l,k} = z_{k,l}$. Then $z := \tilde{z}J_n$ belongs to Ω if it is positive definite. Then, we can calculate the differential operators $\mathrm{d}\pi_\lambda(X^\epsilon_{i,j})$ as in the case of $\mathfrak{so}_{2n}$.

(I) Writing $\left[\begin{smallmatrix} a & 0\\ 0 & d\end{smallmatrix}\right] := \exp(-tX^0_{ij})$, we see that

$$\begin{aligned} \exp(-tX^0_{ij}).z &= azd^{-1}\\ &= \sum_{k,l\in[n]} \left(z_{k,l} - t(\delta_{i,k}z_{j,l} + \delta_{i,l}z_{k,j})\right) E^{(n)}_{k,l} J_n + O(t^2) \end{aligned}$$

and

$$(\det d)^s = e^{st\,\mathrm{tr}\left(E^{(n)}_{n+1-j,n+1-i}\right)+O(t^2)},$$

and hence obtain that

$$\begin{aligned} \mathrm{d}\pi_\lambda(X^0_{i,j}) &= s\delta_{i,j} - \sum_{i\leqslant l} z_{j,l}\partial_{i,l} - \sum_{k\leqslant i} z_{k,j}\partial_{k,i}\\ &= s\delta_{i,j} - \sum_{k\in[n]} z_{j,k}\tilde{\partial}_{i,k}. \end{aligned}$$

(II) Writing $\left[\begin{smallmatrix} 1 & b\\ 0 & 1\end{smallmatrix}\right] := \exp(-tX^+_{ij})$, we see that

$$\begin{aligned} \exp(-tX^+_{ij}).z &= z + b\\ &= \left(\tilde{z} - t(E^{(n)}_{i,j} + E^{(n)}_{j,i})\right) J_n\\ &= \sum_{k,l\in[n]} \left(z_{k,l} - t\delta_{i,k}\delta_{j,l} - t\delta_{j,k}\delta_{i,l}\right) E^{(n)}_{k,l} J_n, \end{aligned}$$

and hence obtain that

$$\begin{aligned} \mathrm{d}\pi_\lambda(X^+_{i,j}) &= -\sum_{k\leqslant l}(\delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k})\partial_{k,l}\\ &= -\tilde{\partial}_{i,j}. \end{aligned}$$

(III) Writing $\left[\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right] := \exp(-tX^-_{ij})$, we see that

$$\begin{aligned}
\exp(-tX^-_{ij}).z &= z(1+cz)^{-1} \\
&= z\left(1_n - tJ_n(E^{(n)}_{i,j} + E^{(n)}_{j,i})z\right)^{-1} \\
&= \left(\tilde{z} + t\tilde{z}(E^{(n)}_{i,j} + E^{(n)}_{j,i})\tilde{z}\right) J_n + O(t^2) \\
&= \sum_{k,l} \left(z_{k,l} + t(z_{k,i}z_{j,l} + z_{k,j}z_{i,l})\right) E^{(n)}_{k,l} J_n + O(t^2),
\end{aligned}$$

and

$$\begin{aligned}
\det(1+cz)^s &= \left(\det(1_n - tJ_n(E^{(n)}_{i,j} + E^{(n)}_{j,i})\tilde{z}J_n)\right)^s = e^{-st\,\mathrm{tr}\left((E^{(n)}_{i,j}+E^{(n)}_{j,i})\tilde{z}\right)+O(t^2)} \\
&= e^{-st(z_{j,i}+z_{i,j})+O(t^2)} = e^{-2stz_{i,j}+O(t^2)},
\end{aligned}$$

and hence obtain that

$$\begin{aligned}
\mathrm{d}\pi_\lambda(X^-_{i,j}) &= -2sz_{i,j} - \sum_{k\leqslant l} \left(z_{k,i}z_{j,l} + z_{k,j}z_{i,l}\right)\partial_{k,l} \\
&= -2sz_{i,j} - \sum_{k,l\in[n]} z_{k,i}z_{j,l}\tilde{\partial}_{k,l}.
\end{aligned}$$

This completes the proof. □

Noting that the dual basis of (5.1) is given by

$$(X^0_{i,j})^\vee = X^0_{j,i}, \qquad\qquad (X^\pm_{i,j})^\vee = \begin{cases} X^\mp_{i,j} & (i \neq j), \\ \frac{1}{2}X^\mp_{i,j} & (i = j), \end{cases}$$

let us define $\sigma(\boldsymbol{X})$ by substituting $\tilde{\xi}_{i,j}$ into $\tilde{\partial}_{i,j}$ following the prescription (3.4) as above.

Theorem 5.2. *Let* $u(z) := \exp \sum_{i\leqslant j} z_{i,j}X^+_{i,j} \in U_\Omega$. *Then we have*

$$\mathrm{Ad}(u(z)^{-1})\sigma(\boldsymbol{X}) = s\sum_i X^0_{i,i} - \sum_{i\leqslant j} \tilde{\xi}_{i,j}X^-_{i,j} \tag{5.5}$$

$$= \left[\begin{array}{cccc|cccc}
s & & & & & & & \\
& s & & & & & & \\
& & \ddots & & & & & \\
& & & s & & & & \\
\hline
-\xi_{1,n-1} & -\xi_{2,n-1} & \cdots & -2\xi_{n,n} & -s & & & \\
\vdots & \vdots & \iddots & \vdots & & \ddots & & \\
-\xi_{1,2} & -2\xi_{2,2} & \cdots & -\xi_{2,n-1} & & & -s & \\
-2\xi_{1,2} & -\xi_{1,2} & \cdots & -\xi_{1,n-1} & & & & -s
\end{array}\right]. \tag{5.6}$$

Proof. Again, this is just a simple matrix calculation, and can be shown in the same way as in the case of $\mathfrak{so}_{2n}$. In fact, if we write $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] := \sigma_\lambda(\boldsymbol{X})$, then the summation $\sum_k z_{i,k}\tilde{\xi}_{j,k}$ in $\sigma_\lambda(X^0_{j,i}) = A_{ij}$ equals $-(zC)_{ij}$, from which it follows that

$$A = s1_n + zC.$$

Similarly, the summation $\sum_{k,l} z_{i,k} z_{l,j} \tilde{\xi}_{k,l}$ in $\sigma_\lambda(X^-_{i,j}) = (BJ_n)_{ij}$ equals $(zCzJ_n)_{ij}$, from which it follows that

$$B = -2sz - zCz.$$

Now exactly the same calculation as in the proof of Theorem 4.3 yields the identity to be shown. □

As in the previous section, we rewrite $\sigma(X^0_{i,i})$ and $\sigma(X^-_{i,j})$ using $\gamma_1 = \sum_{k,l \in [n]} z_{k,l} \tilde{\xi}_{k,l}$ and substitute $s - u$ into γ_1 with a new indeterminate u, which we denote by $\widetilde{\sigma}(X^0_{i,i})$ and $\widetilde{\sigma}(X^-_{i,j})$, respectively:

$$\widetilde{\sigma}(X^0_{i,i}) := u + \sum_{\substack{k,l \in [n] \\ l \neq i}} z_{k,l} \tilde{\xi}_{k,l}, \tag{5.7}$$

$$\widetilde{\sigma}(X^-_{i,j}) := -(u+s) z_{i,j} - \sum_{k,l \in [n]} (z_{k,i} z_{j,l} - z_{i,j} z_{k,l}) \tilde{\xi}_{k,l}. \tag{5.8}$$

As for the others, set $\widetilde{\sigma}(X^\epsilon_{i,j}) := \sigma(X^\epsilon_{i,j})$. Then we define $\widetilde{\sigma}(\boldsymbol{X})$ by

$$\widetilde{\sigma}(\boldsymbol{X}) := \sum_{\epsilon; i,j} \widetilde{\sigma}((X^\epsilon_{i,j})^\vee) \otimes X^\epsilon_{i,j}.$$

Theorem 5.3. *Let $u(z) \in U_\Omega$ be as in Theorem 5.2, then we have*

$$\mathrm{Ad}(u(z)^{-1}) \widetilde{\sigma}(\boldsymbol{X}) = (u + \gamma_1) \sum_i X^0_{i,i} - \sum_{i \leqslant j} \tilde{\xi}_{i,j} X^-_{i,j} - (s - (u + \gamma_1)) \sum_{i \leqslant j} z_{i,j} X^+_{i,j} \tag{5.9}$$

$$= \left[\begin{array}{cccc|cccc} u+\gamma_1 & & & & -\tau z_{1,n-1} & \cdots & -\tau z_{1,2} & -\tau z_{1,1} \\ & u+\gamma_1 & & & -\tau z_{2,n-1} & \cdots & -\tau z_{2,2} & -\tau z_{1,2} \\ & & \ddots & & \vdots & \iddots & \vdots & \vdots \\ & & & u+\gamma_1 & -\tau z_{1,2} & \cdots & -\tau z_{2,n-2} & -\tau z_{1,n-1} \\ \hline -\xi_{1,n-1} & -\xi_{2,n-1} & \cdots & -2\xi_{n,n} & -u-\gamma_1 & & & \\ \vdots & \vdots & \iddots & \vdots & & \ddots & & \\ -\xi_{1,2} & -2\xi_{2,2} & \cdots & -\xi_{2,n-1} & & & -u-\gamma_1 & \\ -2\xi_{1,1} & -\xi_{1,2} & \cdots & -\xi_{1,n-1} & & & & -u-\gamma_1 \end{array} \right]. \tag{5.10}$$

Here, we set $\tau := s - (u + \gamma_1)$ in (5.10) *for brevity.*

Proof. The theorem follows from matrix calculation parallel to that in the proof of Theorem 4.4.

In fact, if we write $\left[\begin{smallmatrix} A(u) & B(u) \\ C & D(u) \end{smallmatrix}\right] := \widetilde{\sigma}(\boldsymbol{X})$, then we see that

$$A(u)_{ij} = (u + \sum_{k,l} z_{k,l} \tilde{\xi}_{k,l}) \delta_{i,j} - \sum_{k=1}^n z_{k,i} \tilde{\xi}_{k,j} \tag{5.11}$$

and the second summation in (5.11) equals $-(\tilde{z}\tilde{C})_{ij} = -(zC)_{ij}$, hence obtain that

$$A(u) = (u + \gamma_1) 1_n + zC.$$

Similarly, we see that the summation in $\widetilde{\sigma}(X^-_{i,j}) = \tilde{B}(u)_{ij}$ equals $z_{i,j}\gamma_1 + (\tilde{z}\tilde{C}\tilde{z})_{ij}$ as shown in the proof of Theorem 5.2, hence obtain that

$$B(u) = -(u + s + \gamma_1) z - zCz. \tag{5.12}$$

Again, exactly the same matrix calculation as in the proof of Theorem 4.4 yields the identity to be shown. □

It follows immediately from Theorem 5.3 and the formula (A.2) that determinant of $\widetilde{\sigma}(\boldsymbol{X})$ yields a generating function for $\{\gamma_k\}$.

Corollary 5.4. *Retain the notation above. Then we have the following formula:*

$$\det(\widetilde{\sigma}(\boldsymbol{X})) = (-1)^n \sum_{k=0}^{n} (u+\gamma_1)^{2n-2k}(s-(u+\gamma_1))^k \gamma_k. \tag{5.13}$$

6. THE CASE $G = \mathrm{SU}(p,q)$ WITH $p \geqslant q$

Finally, we consider the case where $G = \mathrm{SU}(p,q)$, or $\mathfrak{g} = \mathfrak{sl}_{p+q}$, with $p \geqslant q$. In this section, let $E_{i,j}$ denote the $(p+q)\times(p+q)$ matrix with its (i,j)-th entry being 1 and all the others 0. Write an element $V = \mathfrak{u}$ as $Z = \sum_{i\in[p], j\in[q]} z_{i,j}E_{i,j}$. Let us denote by M, D $p\times q$ matrices whose (i,j)-th entries are given by the multiplication operators $z_{i,j}$ and the differential operators $\partial_{i,j}$. Then

$$\mathit{\Gamma}_k := \sum_{\substack{I\subset[p], J\subset[q]\\ |I|=|J|=k}} \det(z^I_J)\det(\partial^I_J) \quad (k = 0,1,\dots,q)$$

form a generating system for $\mathscr{P}\mathscr{D}(V)^{\tilde{K}_{\mathbb{C}}}$ with $\tilde{K}_{\mathbb{C}} = \mathrm{GL}_p\times\mathrm{GL}_q \simeq K_{\mathbb{C}}\times\mathbb{C}^\times$, where z^I_J, ∂^I_J denote submatrices of M, D consisting of the entries whose row- and column- indices are in I and J, respectively ([HU91]).

Take a basis $\{H_i, E^\pm_{i,j}, X^\pm_{i,j}\}$ for $\mathfrak{sl}_{p+q}$ as follows:

$$\begin{aligned}
H_i &:= E_{i,i} - E_{p+q,p+q} && (1\leqslant i\leqslant p+q-1)\\
E^+_{i,j} &:= E_{i,j} && (1\leqslant i\neq j\leqslant p)\\
E^-_{i,j} &:= E_{p+i,p+j} && (1\leqslant i\neq j\leqslant q)\\
X^+_{i,j} &:= E_{i,p+j} && (1\leqslant i\leqslant p, 1\leqslant j\leqslant q)\\
X^-_{i,j} &:= E_{p+j,i} && (1\leqslant i\leqslant p, 1\leqslant j\leqslant q)
\end{aligned} \tag{6.1}$$

Proposition 6.1. *The differential operators* $\mathrm{d}\pi_\lambda(H_i), \mathrm{d}\pi_\lambda(E^\pm_{i,j}), \mathrm{d}\pi_\lambda(X^\pm_{i,j})$ *are given by*

$$\mathrm{d}\pi_\lambda(H_i) = s - \sum_{l=1}^{q} z_{i,l}\partial_{i,l} - \sum_{k=1}^{p} z_{k,q}\partial_{k,q} \qquad (1\leqslant i\leqslant p) \tag{6.2}$$

$$\mathrm{d}\pi_\lambda(H_{p+j}) = \sum_{k=1}^{p}(z_{k,j}\partial_{k,j} - z_{k,q}\partial_{k,q}) \qquad (1\leqslant j\leqslant q-1) \tag{6.3}$$

$$\mathrm{d}\pi_\lambda(E^+_{i,j}) = -\sum_{l=1}^{q} z_{j,l}\partial_{i,l} \qquad (1\leqslant i\neq j\leqslant p) \tag{6.4}$$

$$\mathrm{d}\pi_\lambda(E^-_{i,j}) = \sum_{k=1}^{p} z_{k,i}\partial_{k,j} \qquad (1\leqslant i\neq j\leqslant q) \tag{6.5}$$

$$\mathrm{d}\pi_\lambda(X^+_{i,j}) = -\partial_{i,j} \qquad (1\leqslant i\leqslant p, 1\leqslant j\leqslant q) \tag{6.6}$$

$$\mathrm{d}\pi_\lambda(X^-_{i,j}) = -sz_{i,j} + \sum_{\substack{1\leqslant k\leqslant p,\\ 1\leqslant l\leqslant q}} z_{k,j}z_{i,l}\partial_{k,l}. \qquad (1\leqslant i\leqslant p, 1\leqslant j\leqslant q) \tag{6.7}$$

Proof. Let us denote the $p\times q$ matrix whose (i,j)-th entry is 1 and all the others are 0 by $E^{(p,q)}_{i,j}$, and let $z := \sum_{i\in[p],j\in[q]} z_{i,j}E^{(p,q)}_{i,j}$. Then z belongs to Ω if it is positive definite.

(I) Write $\left[\begin{smallmatrix} a & 0\\ 0 & d\end{smallmatrix}\right] := \exp(-tH_i)$. For $i = 1,\dots,p$, we see that

$$\begin{aligned}\exp(-tH_i).z &= azd^{-1}\\ &= \left(1_p - tE^{(p)}_{i,i} + O(t^2)\right)z\left(1_q + tE^{(q)}_{q,q} + O(t^2)\right)^{-1}\\ &= \sum_{k\in[p],l\in[q]}\left(z_{k,l} - t(\delta_{i,k}z_{i,l} + \delta_{l,q}z_{k,q})\right)E^{(p,q)}_{k,l} + O(t^2)\end{aligned}$$

and

$$(\det d)^s = e^{st\,\mathrm{tr}\left(E^{(q)}_{p+q,p+q}\right)+O(t^2)},$$

and hence obtain that

$$\mathrm{d}\pi_\lambda(H_i) = s - \sum_{l\in[q]} z_{i,l}\partial_{i,l} - \sum_{k\in[p]} z_{k,q}\partial_{k,q} \quad (i = 1,\dots,p).$$

For $j = 1,\dots,q-1$, we see that

$$\begin{aligned}\exp(-tH_{p+j}).z &= azd^{-1}\\ &= z\left(1_q - t(E^{(q)}_{j,j} - E^{(q)}_{q,q}) + O(t^2)\right)^{-1}\\ &= \sum_{k\in[p],l\in[q]}\left(z_{k,l} + t(\delta_{l,j}z_{k,j} - \delta_{l,q}z_{k,q})\right)E^{(p,q)}_{k,l} + O(t^2)\end{aligned}$$

and $(\det d)^s = 1$, hence obtain that

$$\mathrm{d}\pi_\lambda(H_{p+j}) = \sum_{k\in[p]}(z_{k,j}\partial_{k,j} - z_{k,q}\partial_{k,q}) \quad (j = 1,\dots,q-1).$$

(II) Writing $\left[\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right] := \exp(-tE^+_{ij})$, we see that

$$\begin{aligned}\exp(-tE^+_{i,j}).z &= az\\ &= \left(1_p - tE^{(p)}_{i,j}\right)z\\ &= \sum_{k\in[p],l\in[q]}\left(z_{k,l} - t\delta_{i,k}z_{j,l}\right)E^{(p,q)}_{k,l} + O(t^2)\end{aligned}$$

and hence obtain that

$$\mathrm{d}\pi_\lambda(E^+_{i,j}) = -\sum_{l\in[q]} z_{j,l}\partial_{i,l}.$$

(III) Writing $\left[\begin{smallmatrix} 1 & 0\\ 0 & d\end{smallmatrix}\right] := \exp(-tE^-_{ij})$, we see that

$$\begin{aligned}\exp(-tE^-_{i,j}).z &= zd^{-1}\\ &= z\left(1_q - tE^{(q)}_{i,j}\right)^{-1}\\ &= \sum_{k\in[p],l\in[q]}\left(z_{k,l} + t\delta_{j,l}z_{k,i}\right)E^{(p,q)}_{k,l} + O(t^2)\end{aligned}$$

and $\det d = 1$, and hence obtain that

$$\mathrm{d}\pi_\lambda(E^-_{i,j}) = \sum_{k\in[p]} z_{k,i}\partial_{k,j}.$$

(IV) Writing $\left[\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right] := \exp(-tX^+_{ij})$, we see that

$$\begin{aligned}\exp(-tX^+_{ij}).z &= z + b\\ &= z - tE^{(p,q)}_{i,j}\end{aligned}$$

and hence obtain that

$$\mathrm{d}\pi_\lambda(X^+_{i,j}) = -\partial_{i,j}.$$

(V) Writing $\left[\begin{smallmatrix}1 & 0\\ c & 1\end{smallmatrix}\right] := \exp(-tX^-_{ij})$, we see that

$$\begin{aligned}\exp(-tX^-_{ij}).z &= z(1+cz)^{-1}\\ &= z\left(1_q - tE^{(q,p)}_{j,i}z\right)^{-1}\\ &= z + tzE^{(q,p)}_{j,i}z + O(t^2)\\ &= \sum_{k\in[p],l\in[q]}\left(z_{k,l} + tz_{k,j}z_{i,l}\right)E^{(p,q)}_{k,l} + O(t^2),\end{aligned}$$

and

$$\begin{aligned}\det(1+cz)^s &= \left(\det(1_q - tE^{(q,p)}_{j,i}z)\right)^s = e^{-st\,\mathrm{tr}\left(E^{(q,p)}_{j,i}z\right)+O(t^2)}\\ &= e^{-stz_{i,j}+O(t^2)},\end{aligned}$$

and hence obtain that

$$\mathrm{d}\pi_\lambda(X^-_{i,j}) = -sz_{i,j} - \sum_{k\in[p],l\in[q]} z_{k,j}z_{i,l}\partial_{k,l}.$$

This completes the proof. □

Noting that the dual basis of (6.1) is given by

$$H_i^\vee = H_i - \tfrac{1}{p+q}\sum_{k=1}^{p+q-1} H_k, \qquad (E^\pm_{i,j})^\vee = E^\pm_{j,i}, \qquad (X^\pm_{i,j})^\vee = X^\mp_{i,j},$$

let us define $\sigma(\boldsymbol{X})$ following the prescription (3.4) as above.

Theorem 6.2. *Let* $u(z) := \exp \sum_{i\in[p],j\in[q]} z_{i,j}X^+_{i,j} \in U_\Omega$ *Then we have*

$$\mathrm{Ad}(u(z)^{-1})\sigma(\boldsymbol{X}) = \tfrac{p}{p+q}s\sum_{i=1}^{p} H_i - \tfrac{q}{p+q}s\sum_{j=1}^{q-1} H_{p+j} - \sum_{i\leqslant j}\xi_{i,j}X^-_{i,j} \tag{6.8}$$

$$= \left[\begin{array}{cccc|cccc} \frac{p}{p+q}s & & & & & & & \\ & \frac{p}{p+q}s & & & & & & \\ & & \ddots & & & & & \\ & & & \frac{p}{p+q}s & & & & \\ \hline -\xi_{1,1} & -\xi_{2,1} & \cdots & -\xi_{p,1} & -\frac{q}{p+q}s & & & \\ -\xi_{1,2} & -\xi_{2,2} & \cdots & -\xi_{p,2} & & -\frac{q}{p+q}s & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ -\xi_{1,q} & -\xi_{2,q} & \cdots & -\xi_{p,q} & & & & -\frac{q}{p+q}s \end{array}\right]. \tag{6.9}$$

Proof. It follows from (6.2) and (6.3) that

$$\frac{1}{p+q}\sum_{k=1}^{p+q-1}\sigma(H_k) = \frac{p}{p+q}s - \sum_{k\in[p]} z_{k,q}\xi_{k,q}.$$

Thus we obtain that

$$\begin{aligned} \sigma(H_i^\vee) &= \frac{q}{p+q}s - \sum_{l\in[q]} z_{i,l}\xi_{i,l} \qquad (i = 1,\dots,p) \\ \sigma(H_{p+j}^\vee) &= -\frac{p}{p+q}s + \sum_{k\in[p]} z_{k,j}\xi_{k,j} \quad (j = 1,\dots,q-1). \end{aligned} \tag{6.10}$$

Now, if we write $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] := \sigma_\lambda(\boldsymbol{X})$, then

$$A_{ij} = \frac{q}{p+q}s\delta_{i,j} - \sum_{l\in[q]} z_{i,l}\xi_{j,l} \qquad (i,j\in[p]) \tag{6.11a}$$

$$B_{ij} = -sz_{i,j} + \sum_{k\in[p],j\in[q]} z_{k,j}z_{i,l}\xi_{k,l} \qquad (i\in[p],\ j\in[q]) \tag{6.11b}$$

$$C_{ij} = -\xi_{j,i} \qquad (i\in[p],\ j\in[q]) \tag{6.11c}$$

$$D_{ij} = -\frac{p}{p+q}s\delta_{i,j} + \sum_{k\in[p]} z_{k,j}\xi_{k,i} \qquad (i,j\in[q]) \tag{6.11d}$$

by (6.4), (6.5), (6.6), (6.7) and (6.10). Now exactly the same matrix calculation as in the proof of Theorem 5.2 yields the formula to be shown. □

As in the previous sections, we rewrite $\sigma(H_i^\vee)$ and $\sigma(X^-_{i,j})$ using $\gamma_1 = \sum_{k\in[p],l\in[q]} z_{k,l}\xi_{k,l}$ and substitute $s-u$ into γ_1 with a new indeterminate u, which we denote by $\widetilde{\sigma}(H_i^\vee)$ and

$\widetilde{\sigma}(X^-_{i,j})$, respectively:

$$\widetilde{\sigma}(H^\vee_i) := (u - \frac{p}{p+q}s) + \sum_{\substack{k\in[p],l\in[q]\\ k\neq i}} z_{k,l}\xi_{k,l} \qquad (i = 1,\dots,p) \tag{6.12}$$

$$\widetilde{\sigma}(H^\vee_{p+j}) := (-u + \frac{q}{p+q}s) + \sum_{\substack{k\in[p],l\in[q]\\ l\neq j}} z_{k,l}\xi_{k,l} \qquad (j = 1,\dots,q-1) \tag{6.13}$$

$$\widetilde{\sigma}(X^-_{i,j}) := -uz_{i,j} - \sum_{\substack{k\in[p],l\in[q]\\ k\neq i,l\neq j}} (z_{i,j}z_{k,l} - z_{i,l}z_{k,j})\xi_{k,l}. \quad (i = 1,\dots,p;\ j = 1,\dots,q) \tag{6.14}$$

As for the others, set $\widetilde{\sigma}(\cdot) := \sigma(\cdot)$. Then we define $\widetilde{\sigma}(\boldsymbol{X})$ by

$$\widetilde{\sigma}(\boldsymbol{X}) = \sum_i \widetilde{\sigma}(H^\vee_i)\otimes H_i + \sum_{\epsilon;i,j}\left(\widetilde{\sigma}((E^\epsilon_{i,j})^\vee)\otimes E^\epsilon_{i,j} + \widetilde{\sigma}((X^\epsilon_{i,j})^\vee)\otimes X^\epsilon_{i,j}\right).$$

Theorem 6.3. *Let $u(z)\in U_\Omega$ be as in Theorem 6.2, then we have*

$$\begin{aligned}\mathrm{Ad}(u(z)^{-1})\widetilde{\sigma}(\boldsymbol{X}) &= \left(u+\gamma_1 - \tfrac{p}{p+q}s\right)\sum_{i=1}^{p} H_i + \left(-u-\gamma_1+\tfrac{q}{p+q}s\right)\sum_{j=1}^{q-1} H_{p+j}\\ &\quad - \sum_{i,j}\xi_{i,j}X^-_{i,j} - (s-(u+\gamma_1))\sum_{i,j} z_{i,j}X^+_{i,j}\end{aligned} \tag{6.15}$$

$$= \left[\begin{array}{cccc|cccc} u_+ & & & & -\tau z_{1,1} & -\tau z_{2,1} & \cdots & -\tau z_{1,q}\\ & u_+ & & & -\tau z_{2,1} & -\tau z_{2,2} & \cdots & -\tau z_{2,q}\\ & & \ddots & & \vdots & \vdots & & \vdots\\ & & & u_+ & -\tau z_{p,1} & -\tau z_{p,2} & \cdots & -\tau z_{p,q}\\ \hline -\xi_{1,1} & -\xi_{2,1} & \cdots & -\xi_{p,1} & u_- & & & \\ -\xi_{1,2} & -\xi_{2,2} & \cdots & -\xi_{p,2} & & u_- & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ -\xi_{1,q} & -\xi_{2,q} & \cdots & -\xi_{p,q} & & & & u_- \end{array}\right]. \tag{6.16}$$

Here, we set $u_+ := u + \gamma_1 - \frac{p}{p+q}s$, $u_- := -u - \gamma_1 + \frac{q}{p+q}s$ and $\tau := s - (u+\gamma_1)$ in (6.16) *for brevity.*

Proof. If we write $\left[\begin{smallmatrix} A(u) & B(u)\\ C & D(u)\end{smallmatrix}\right] := \widetilde{\sigma}(\boldsymbol{X})$, then we can show that

$$A(u) = (u+\gamma_1 - \tfrac{p}{p+q}s)1_p + zC \tag{6.17}$$

$$B(u) = -(u+\gamma_1)z - zCz \tag{6.18}$$

$$D(u) = (-u-\gamma_1+\tfrac{q}{p+q}s)1_q - Cz \tag{6.19}$$

exactly in the same way as in the proof of Theorems 4.4 and 5.3, and hence obtain the theorem. □

It follows immediately from Theorem 6.3 and the formula (A.2) that determinant of $\widetilde{\sigma}(\boldsymbol{X})$ yields a generating function for $\{\gamma_k\}$.

Corollary 6.4. *Retain the notation above. Then we have the following formula:*

$$\det(\widetilde{\sigma}(\boldsymbol{X})) = (-1)^q \sum_{k=0}^{q} (u+\gamma_1 - \tfrac{p}{p+q}s)^{p-k}(u+\gamma_1 - \tfrac{q}{p+q}s)^{q-k}(s-(u+\gamma_1))^k \gamma_k. \tag{6.20}$$

7. Concluding Remark

In general, let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. If M is a G-manifold, then the cotangent bundle T^*M is a symplectic G-manifold, and the moment map $\mu : T^*M \to \mathfrak{g}^*$ is given by

$$\langle \mu(x,\xi), X\rangle = \xi(X_M(x)) \qquad (x \in M, \xi \in T_x^*M) \tag{7.1}$$

for $X \in \mathfrak{g}$, where $\langle \cdot,\cdot\rangle$ denotes the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and $X_M(x) \in T_xM$ the tangent vector at $x \in M$ defined by

$$X_M(x)f := \frac{d}{dt}\Big|_{t=0} f(\exp(-tX).x) \tag{7.2}$$

for functions f defined around $x \in M$ (see e.g. [MFK94]).

Returning to our case, if the character λ is trivial, it follows from (7.1) and (7.2) that $\sigma(\boldsymbol{X})$ is identical to the moment map $\mu : T^*(G_{\mathbb{C}}/P) \to \mathfrak{g}^*$ composed by the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ via Killing form (or the $G_{\mathbb{C}}$-invariant nondegenerate bilinear form B given in (3.2); our Lie algebras are simple!), which we also denote by μ.

If λ is nontrivial, then $\sigma(\boldsymbol{X})$ can be regarded as a variant of the twisted moment map $\mu_\lambda : T^*(G_{\mathbb{C}}/P) \to \mathfrak{g}^* \simeq \mathfrak{g}$ ([SV96]); the difference $\mu_\lambda - \mu$, which they denote by λ_x with $x \in G_{\mathbb{C}}/P$ therein, can be expressed as $\lambda_x = \mathrm{Ad}(g)\lambda^\vee$, where $\lambda^\vee \in \mathfrak{g}$ corresponds to $\lambda \in \mathfrak{g}^*$ via the nondegenerate bilinear form B and g is an element of a compact real form $U_{\mathbb{R}}$ of $G_{\mathbb{C}}$ such that $x = g\dot{e}$ with $\dot{e}$ the origin P of $G_{\mathbb{C}}/P$. Note that if x is in the open subset $G/K = GP/P = U_\Omega P/P \subset G_{\mathbb{C}}/P$, one can choose u_x from U_Ω so that $x = u_x\dot{e}$ instead of g from $U_{\mathbb{R}}$, which was denoted by $u(z)$ in §§4-6. Then it is immediate to verify that

$$\sigma(\boldsymbol{X}) = \mathrm{Ad}(u_x)\lambda^\vee + \mu(x,\xi).$$

Writing $\mu'_\lambda(x,\xi) := \sigma(\boldsymbol{X})$ to make its dependence on x and ξ transparent, Theorems 4.3, 5.2, and 6.2 state that an analogue of $U_{\mathbb{R}}$-equivariance of the twisted moment map holds:

$$\mathrm{Ad}(u_x^{-1})\mu'_\lambda(x,\xi) = \mu'_\lambda(u_x^{-1}.(x,\xi)) \tag{7.3}$$

with $x = u_x\dot{e}$, though U_Ω is not a subgroup. Since $u_x^{-1}.(x,\xi) = (\dot{e},\xi)$, the relation (7.3) says that all the principal symbols of the representation operators of elements of the Lie algebras are determined by those at the origin.

At this moment, the following question naturally arise: *What is the geometric meaning of the indeterminate u introduced in the definition of $\widetilde{\sigma}(\boldsymbol{X})$?*

We will seek for the answer to this question in the forthcoming paper.

Acknowledgments I would like to thank Tôru Umeda for guiding me to the world of the Capelli identity, as well as for his valuable comments and advice.

Appendix A. Minor Summation Formulae

In this appendix, we collect formulae concerning Pfaffian and determinant needed to show Corollaries 4.5, 5.4 and 6.4.

Let X be a $2n \times 2n$ matrix alternating along the anti-diagonal. Since XJ_{2n} is an alternating matrix, one can define Pfaffian of XJ_{2n}, which is denoted by $\mathrm{Pf}\,(X)$ as mentioned above. If we write X as $X = \left[\begin{smallmatrix} a & b \\ c & -J_n {}^t a J_n \end{smallmatrix}\right]$ with submatrices a, b, c all being of size $n \times n$, then b and c are also alternating along the anti-diagonal. Thus we parametrize the submatrices a, b, c as follows:

$$a = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}, \quad b = \begin{bmatrix} b_{1,n} & \cdots & b_{1,2} & 0 \\ \vdots & \iddots & 0 & -b_{1,2} \\ b_{n-1,n} & 0 & \iddots & \vdots \\ 0 & -b_{n-1,n} & \cdots & -b_{1,n} \end{bmatrix}, \quad c = \begin{bmatrix} c_{1,n} & \cdots & c_{n-1,n} & 0 \\ \vdots & \iddots & 0 & -c_{n-1,n} \\ c_{1,2} & 0 & \iddots & \vdots \\ 0 & -c_{1,2} & \cdots & -c_{1,n} \end{bmatrix}.$$

Theorem A.1 ([IW06])**.** *Let $X = \left[\begin{smallmatrix} a & b \\ c & -J_n {}^t a J_n \end{smallmatrix}\right]$ be a $2n \times 2n$ matrix alternating along the anti-diagonal with submatrices a, b, c parametrized as above. Then we can expand Pfaffian* $\mathrm{Pf}\,(X)$ *in the following way.*

$$\mathrm{Pf}\,(X) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{I,J \subset [n] \\ |I|=|J|=2k}} \mathrm{sgn}\left(\bar{I}, I\right) \mathrm{sgn}\left(\bar{J}, J\right) \det(a^{\bar{I}}_{\bar{J}})\, \mathrm{Pf}\,(b_I)\, \mathrm{Pf}\,(c_J)\,. \tag{A.1}$$

Here, for $I, J \subset [n]$, $\bar{I}$ denotes the complement of I in $[n]$, a^I_J the submatrix of a whose row- and column-indices are in I and J respectively, b_I, c_I the submatrices of b, c whose row- and column-indices are both in I, and $\mathrm{sgn}\,(I, J)$ *the signature of the permutation* $\left(\begin{smallmatrix} 1,2,\cdots,n \\ I \;\; J \end{smallmatrix}\right)$.

In order to show Corollaries 5.4 and 6.4, it suffices to consider a square matrix X of the form $X = \left[\begin{smallmatrix} u1_p & b \\ c & v1_q \end{smallmatrix}\right]$ with $u, v \in \mathbb{C}$ and submatrices b, c of size $p \times q$, $q \times p$ respectively. Let us parametrize X as follows:

$$X = \left[\begin{array}{cccc|ccc} u & & & & b_{1,1} & \cdots & b_{1,q} \\ & u & & & b_{2,1} & \cdots & b_{2,q} \\ & & \ddots & & \vdots & & \vdots \\ & & & u & b_{p,1} & \cdots & b_{p,q} \\ \hline c_{1,1} & c_{2,1} & \cdots & c_{p,1} & v & & \\ \vdots & \vdots & & \vdots & & \ddots & \\ c_{1,q} & c_{2,q} & \cdots & c_{p,q} & & & v \end{array}\right].$$

Proposition A.2. *Let X be a square matrix given as above. Then we can expand* $\det X$ *in the following way:*

$$\det X = \sum_{k=0}^{q} \sum_{\substack{I \subset [p], J \subset [q] \\ |I|=|J|=k}} u^{p-k} v^{q-k} \det(b^I_J) \det(c^I_J). \tag{A.2}$$

REFERENCES

[MFK94] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory (3rd enlarged edition)*, Springer-Verlag, 1994.

[Hasa] T. Hashimoto, *A central element in the universal enveloping algebra of type* D_n *via minor summation formula of Pfaffians*, arXiv:math.RT/0602055.

[Hasb] ———, *Generating function for* GL_n*-invariant diffrential operators in the skew Capelli identity*, arXiv:0803.1339 [math.RT].

[HOOS93] T. Hashimoto, K. Ogura, K. Okamoto, and R. Sawae, *Borel-Weil theory and Feynman path integrals on flag manifolds*, Hiroshima Math. J. **23** (1993), 231–247.

[HU91] R. Howe and T. Umeda, *The Capelli identity, the double commutant theorem, and multiplicity-free actions*, Math. Ann. **290** (1991), 565–619.

[Ito01] M. Itoh, *A Cayley-Hamilton theorem for the skew Capelli elements*, J. Algebra. **242** (2001), 740–761.

[IU01] M. Itoh and T. Umeda, *On central elements in the universal enveloping algebras of the orthogonal Lie algebra*, Compositio Math. **127** (2001), 333–359.

[IW06] M. Ishikawa and M. Wakayama, *Application of minor summation formula III, Plücker relations, lattice paths and Pfaffian identities*, J. Comb. Theory A **113** (2006), 113–155.

[Kna86] A. W. Knapp, *Representation theory of semisimple groups: An overview based on examples*, Princeton Mathematical Series, vol. 36, Princeton Univ. Press, 1986.

[KW02] K. Kinoshita and M. Wakayama, *Explicit Capelli identities for skew symmetric matrices*, Proc. Edinburgh Math. Soc. **45** (2002), 449–465.

[Shi84] G. Shimura, *On differential operators attached to certain representations of classical groups*, Invent. Math. **77** (1984), 463–488.

[SV96] W. Schmid and K. Vilonen, *Characteristic cycles of constructible sheaves*, Invent. Math. **124** (1996), 451–502.

DEPARTMENT OF INFORMATICS, GRADUATE SCHOOL OF ENGINEERING, TOTTORI UNIVERSITY, 4-101, KOYAMA-MINAMI, TOTTORI, 680-8552, JAPAN

E-mail address: `thashi@ike.tottori-u.ac.jp`